Computing Dynamic User Equilibrium on large scale networks without knowing global paramters

Duong Viet Thong[1] Aviv Gibali[2] Mathias Staudigl[3]
Phan Tu Vuong[4]

[1]Thu Dau Mot University, [2]ORT Braude College [3]Maastricht University [4]University of Southampton

Maastricht University

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Dynamic User equilibrium

- Dynamic User equilibrium of open-loop type is one type of DTA
- Mathematical formulation of Wardrop's first principle: In an equilibrium all used paths have the same effective unit travel delay.
- DUE models consist of 5 main components:
  1. A model of path delay;
  2. flow dynamics;
  3. flow propagation constraints;
  4. route and departure time choice model;
  5. model of demand growth.

Our Contribution

This work focuses on advancing the computational aspects of DUE.

- Strong Convergence of path flows to a DUE.
- Adaptive Step size selection: No information about the Lipschitz coefficient of the delay operator needed.
- Weak Monotonicity: Algorithm is provably convergent under pseudo-monotonicity.
- Inertial and relaxation effects lead to enhanced stability and potential acceleration.

Network Model

- $G = (\mathcal{V}, \mathcal{E})$ a directed graph, $\mathcal{V}$ set of vertices representing junctions and $\mathcal{E}$ representing road segments.
- Origin-Destination (O-D) pairs $w_i = (o_i, d_i) \in \mathcal{W}$.
- A path $p = \{l_1, \ldots, l_{m(p)}\}$ is a non-repeating set of links connected some O-D pair $w \in \mathcal{W}$.
- Let $\mathcal{P}$ be the set of paths and $\mathcal{H} = \mathbb{R}^{|\mathcal{P}|}$, and $\mathcal{P}_w \subset \mathcal{P}$ paths connecting the pair $w \in \mathcal{W}$.
- $(\forall w \in \mathcal{W}) : Q_w > 0$ exogenous demand.

Paths and Delays

- $[t_0, t_1]$ for $0 < t_0 < t_1 < \infty$ is a fixed planning horizon.
- $h = (h_p)_{p \in \mathcal{P}}$ is a profile of *departure time-densities* (path flow)
- $h_p \in L^2_+[t_0, t_1]$, so that $h \in \mathcal{H} = \left(L^2_+[t_0, t_1]\right)^{|\mathcal{P}|}$.
- $D_p(t, h)$ for $p \in \mathcal{P}$, $t \in [t_0, t_1]$ the path travel time of a driver departing at time t and following path p
- The *path delay operator*

$$D(h) = \{D_p(\cdot, h); p \in \mathcal{P}\} : \mathcal{H} \to \mathcal{H}$$
$$h \mapsto D(h)$$

Effective delay

- Beside path travel times, we also allow for arrival penalties
$$t \mapsto \phi(t + D_p(t,h) - \tau)$$
- This gives rise to the *effective delay operator*
$$A_p(t,h) = D_p(t,h) + \phi(t - D_p(t,h) - \tau).$$
- The effective path delay operator is a key component of analytical DUE models.
- It is usually not available in closed form.
- Numerical evaluation from *dynamic network loading* (DNL).

Definition of DUE

Let

$$\mathcal{X} = \{h \in \mathcal{H} | (\forall w \in \mathcal{W}) : \sum_{p \in \mathcal{P}_w} \int_{t_0}^{t_1} h_p(t) \mathrm{d}t = Q_w\}$$

the set of feasible flows.

Definition

$h^* \in \mathcal{H}$ is a DUE if

(a) $h^* \in \mathcal{X}$,

(b) $h_p^*(t) > 0, p \in \mathcal{P}_w \Rightarrow A_p(t, h^*) = v_w(h^*)$

where

$$v_p(h) = \operatorname*{ess\,inf}_{t \in [t_0, t_1]} A_p(t, h), \; v_w(h) = \min_{p \in \mathcal{P}_w} v_p(h).$$

VI Formulation

$\mathcal{H}$ is a real Hilbert space with inner product

$$\langle f, g \rangle = \int_{t_0}^{t_1} \sum_{p \in \mathcal{P}} f_p(t) g_p(t) \mathrm{d}t.$$

Proposition (Friesz, Bernstein, Smith, Tobin and Wie, 1993)

$h^* \in \mathcal{X}$ is a DUE with simultaneous route-and-departure-time choice if

$$(\forall h \in \mathcal{X}) : \langle A(h^*), h - h^* \rangle \geq 0$$

Opens the door to apply fixed-point algorithms to DUE!

Fixed point formulation of DUE

- Define the residual function

$$r_\lambda(h) = \|h - P_{\mathcal{X}}(h - \lambda A(h))\|$$

- Given $\lambda > 0$ on can see that $h \in \mathcal{X}$ is a DUE if and only if $r_\lambda(h) = 0$.
- This suggests the *fixed-point iteration*

$$h_{n+1} = P_{\mathcal{X}}(h_n - \lambda_n A(h_n))$$

  for $\{\lambda_n\}_n$ step size sequence.
- Weak convergence of $\{h_n\}_n$ follows under monotonicity and co-cocercivty of A.
- It is of great interest to relax this assumption.

Assumptions

Assumption

$A : \mathcal{H} \to \mathcal{H}$ is sequentially weakly continuous

Assumption

$A : \mathcal{H} \to \mathcal{H}$ is pseudo-monotone:

$$\langle A(h_1), h_2 - h_1 \rangle \geq 0 \Rightarrow \langle A(h_2), h_2 - h_1 \rangle \geq 0.$$

for all $h_1, h_2 \in \mathcal{H}$.

Link-based Macroscopic model

The within-link dynamics follow a standard conservation law

$$\partial_t \rho_i(t,x) + \partial_x \left[\rho_i(t,x) v_i(\rho_i(t,x))\right] = 0$$

for $(t,x) \in [t_0, t_1] \times [a_i, b_i]$.
We focus on the triangular fundamental diagram

$$f_i(\rho) = \begin{cases} v_i \rho & \text{if } \rho \in [0, \rho_i^c], \\ -w_i(\rho - \rho^{jam}) & \text{if } \rho \in (\rho_i^c, \rho_i^{jam}] \end{cases}$$

DAE Formulation

- Following the variational formulation of Aubin et al. (2008) and Claudel & Bayen (2010), consider the Moskowitz function $N_i(t,x)$:

$$\partial_t N_i(t,x) = f_i(\rho_i(t,x)), \partial_x N_i(t,x) = -\rho_i(t,x),$$
$$\partial_t N_i(t,x) - f_i(-\partial_x N_i(t,x)) = 0.$$

- Include dynamics at junctions and origin nodes.
- Calculate path delay by the implicit relation

$$N^{up}(t) = N^{down}(\lambda(t)),$$
$$D_p(t,h) = \lambda_s \circ \lambda_1 \circ \ldots \circ \lambda_{m(p)}(t).$$

Basic fixed-point iteration

Algorithm 1 FB for VI$(A, \mathcal{X})$.

Input: Effective delay operator $A : \mathcal{H} \to \mathcal{H}$, step size $\{\tau_n\}_{n\in\mathbb{N}}$, Initial point $h_0 \in \mathcal{X}$, $N \geq 1$ stopping time.
for $n = 0, 1, \ldots, N$ **do**
  obtain h_n by running a DNL procedure;
  if Stopping condition not satisfied **then**
    Update

$$h_{n+1} = P_{\mathcal{X}}(h_n - \tau_n A(h_n)). \quad (1)$$

  end if
end for

- FB is a standard method to solve variational inequalities.
- Weak convergence of path flows requires
  1. Lipschitz continuity: $\|A(h) - A(h')\| \leq L \|h - h'\|$ for some $L > 0$;
  2. Monotonicity: $\langle A(h) - A(h'), h - h' \rangle \geq 0$ for all $h, h' \in \mathcal{H}$;
  3. Co-coercivity: $\langle A(h) - A(h'), h - h' \rangle \geq \beta \|A(h) - A(h')\|^2$ for some $\beta > 0$.
- In DUE problems these assumptions are either not satisfied or hard to justify.

A forward-backward-forward algorithm

Algorithm 2 FBF for VI$(A, \mathcal{X})$.

Input: Initial point $h_0 \in \mathcal{X}$.
for $n = 0, 1, \dots, N$ **do**
  Obtain h_n by running a DNL procedure;
  if Stopping condition not satisfied **then**
    Compute

$$\begin{aligned} y_n &= P_{\mathcal{X}}(h_n - \tau_n A(h_n)), \\ z_{n+1} &= y_n + \tau_n(A(h_n) - A(y_n)), \\ h_{n+1} &= (1 - \alpha_n - \beta_n)h_n + \beta_n z_{n+1}. \end{aligned}$$

    Update the step size sequence

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\mu\|y_n - h_n\|}{\|A(y_n) - A(h_n)\|}\right\} & \text{if } A(y_n) - A(h_n) \neq 0, \\ \tau_n & \text{else} \end{cases} \tag{2}$$

  end if
end for

Introducing Inertia

Algorithm 3 IFBF for VI$(A, \mathcal{X})$.

Input: Initial point $h_{-1}, h_0 \in \mathcal{X}$.
for $n = 0, 1, \ldots, N$ **do**
obtain h_n via a DNL procedure
if Stopping condition not satisfied **then**
Compute

$$\begin{aligned} w_n &= (1-\beta_n)[h_n + \alpha_n(h_n - h_{n-1})] \\ y_n &= P_{\mathcal{X}}(w_n - \tau_n A(w_n)) \\ h_{n+1} &= (1-\lambda)w_n + \lambda(y_n + \tau_n(A(w_n) - A(y_n))) \end{aligned}$$

Update the step size

$$\tau_{n+1} = \begin{cases} \tau_n & \text{if } A(w_n) = A(y_n) \\ \min\{\tau_n, \frac{\mu\|w_n - y_n\|}{\|A(w_n) - A(y_n)\|}\} & \text{else} \end{cases} \tag{3}$$

Update the inertia parameter

$$0 \leq \alpha_{n+1} \leq \begin{cases} \min\left\{\alpha, \frac{\epsilon_{n+1}}{\|h_{n+1} - h_n\|}\right\} & \text{if } h_{n+1} \neq h_n, \\ \alpha & \text{otherwise.} \end{cases} \tag{4}$$

Strong convergence of FBF

Theorem (Theorem 2, Duvocelle, Meier, S., Vuong, 2020)

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two real sequences in $(0,1)$, satisfying conditions

$$\{\beta_n\}_{n\in\mathbb{N}} \subseteq (b, 1-\alpha_n) \text{ for some } b > 0, \tag{5}$$

and

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{6}$$

Then the sequence $\{h_n\}$ generated by Algorithm 2 converges strongly to $h^* = \arg\min\{\|z\| : z \in \Omega\}$.

Strong convergence of IFBF

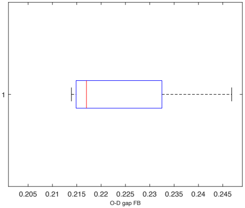
Theorem (Viet Thong, Gibali, S., Vuong, 2021)

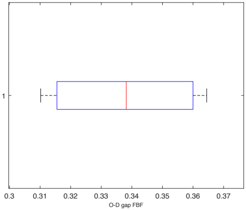
Let $\{\varepsilon_n\}_n$ *and* $\{\beta_n\}_n$ *be such that*

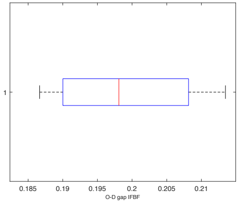
$$\lim_{n\to\infty}\beta_n = 0, \sum_n \beta_n = \infty, \text{ and } \lim_{n\to\infty}\frac{\varepsilon_n}{\beta_n} = 0.$$

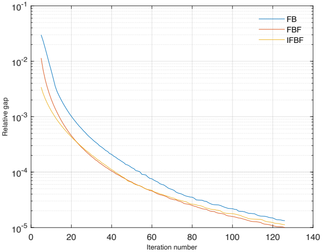
Let $\{h_n\}$ *be generated by Algorithm 3. Then* $h_n \to h^* = \operatorname{argmin}_{z\in\Omega}\|z\|$.

Box Plot for the Gap









Box Plot for the Gap